Najkrótsze przedziały ufności Cloppera–Pearsona

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Model statystyczny

$$(\{0,1,\ldots,n\},\{Bin(n,\pi), 0<\pi<1\}),$$

$$\binom{n}{x}\pi^x(1-\pi)^{n-x}, \quad x=0,1,\ldots,n.$$

Przedział ufności dla prawdopodobieństwa π na poziomie ufności γ

$$\left(F^{-1}\left(X, n-X+1; \frac{1-\gamma}{2}\right); F^{-1}\left(X+1, n-X; \frac{1+\gamma}{2}\right)\right),$$

Dla $X=0$ lewy koniec $=0$

Dla $X=n$ prawy koniec $=1$

$$\left(F^{-1}(X, n - X + 1; \gamma_1); F^{-1}(X + 1, n - X; \gamma_2)\right),$$

$$d(\gamma_1, x) = F^{-1}(x + 1, n - x; \gamma + \gamma_1) - F^{-1}(x, n - x + 1; \gamma_1).$$

Dla danego x znaleźć $0 < \gamma_1 < 1 - \gamma$ takie, że $d(\gamma_1, x)$ jest najmniejsze.

$$\frac{\partial d(\gamma_1, x)}{\partial \gamma_1} =$$
$$B(x+1, n-x)(1-F^{-1}(x+1, n-x; \gamma+\gamma_1))^{1+x-n} F^{-1}(x+1, n-x; \gamma+\gamma_1)^{-x}$$
$$- B(x, n-x+1)(1-F^{-1}(x, n-x+1; \gamma_1))^{x-n} F^{-1}(x, n-x+1; \gamma_1)^{1-x}.$$

Niech

$$LHS(\gamma_1, x) = \frac{[1-F^{-1}(x+1, n-x; \gamma+\gamma_1)]^{n-x-1} F^{-1}(x+1, n-x; \gamma+\gamma_1)^{x}}{B(x+1, n-x)}$$
$$RHS(\gamma_1, x) = \frac{[1-F^{-1}(x, n-x+1; \gamma_1)]^{n-x} F^{-1}(x, n-x+1; \gamma_1)^{x-1}}{B(x, n-x+1)}$$

Wówczas

$$\frac{\partial d(\gamma_1, x)}{\partial \gamma_1} = \frac{1}{LHS(\gamma_1, x)} - \frac{1}{RHS(\gamma_1, x)}.$$

$$F^{-1}(x, n-x+1; 0) = 0 \quad \text{oraz} \quad F^{-1}(x+1, n-x; 1) = 1,$$

Dla $1 < x < n-1$

$$\gamma_1 \to 0 \Rightarrow LHS(\gamma_1, x) > 0 \text{ oraz } RHS(\gamma_1, x) \to 0^+,$$
$$\gamma_1 \to 1-\gamma \Rightarrow LHS(\gamma_1, x) \to 0^+ \text{ oraz } RHS(\gamma_1, x) > 0.$$

Równanie

$$\frac{\partial d(\gamma_1, x)}{\partial \gamma_1} = 0 \qquad (*)$$

ma rozwiązanie

$LHS(\cdot, x)$ i $RHS(\cdot, x)$ są jednomodalne i wklęsłe na przedziale $(0, 1 - \gamma)$

Równanie $(*)$ ma jedno rozwiązanie

Ciekawostka: dla parzystych n oraz $x = n/2$

$$\gamma_1^* = (1 - \gamma)/2$$

Dla $x = 1$ mamy

$$LHS(\gamma_1, 1) = \frac{[1 - F^{-1}(2, n-1; \gamma + \gamma_1)]^{n-2} F^{-1}(2, n-1; \gamma + \gamma_1)}{B(2, n-1)},$$

$$RHS(\gamma_1, 1) = \frac{[1 - F^{-1}(1, n; \gamma_1)]^{n-1}}{B(1, n)} = n(1 - \gamma_1)^{\frac{n-1}{n}}.$$

$$\gamma_1 \to 0 \Rightarrow \begin{array}{l} LHS(\gamma_1, 1) \to \dfrac{[1 - F^{-1}(2, n-1; \gamma)]^{n-2} F^{-1}(2, n-1; \gamma)}{B(2, n-1)} \\ RHS(\gamma_1, x) \to n, \end{array}$$

$$\gamma_1 \to 1 - \gamma \Rightarrow LHS(\gamma_1, x) \to 0 \text{ oraz } RHS(\gamma_1, x) \to n\gamma^{\frac{n-1}{n}}.$$

$$RHS(\gamma_1, 1) > LHS(\gamma_1, 1) \quad \text{for } 0 < \gamma_1 < 1 - \gamma.$$

Najkrótszy przedział ufności jest jednostronny.

Podobnie dla $x = n - 1$.

$n = 20$

	Najkrótszy				Clopper-Pearson		
x	γ_1^*	*lewy*	*prawy*	*dlugosc*	*lewy*	*prawy*	*dlugosc*
0	0.00000	0.00000	0.13911	0.13911	0.00000	0.16843	0.16843
1	0.00000	0.00000	0.21611	0.21611	0.00127	0.24873	0.24747
2	0.00125	0.00261	0.28393	0.28132	0.01235	0.31698	0.30463
3	0.00561	0.01839	0.34998	0.33159	0.03207	0.37893	0.34686
4	0.00966	0.04318	0.41249	0.36931	0.05733	0.43661	0.37928
5	0.01302	0.07344	0.47156	0.39812	0.08657	0.49105	0.40447
6	0.01587	0.10763	0.52766	0.42004	0.11893	0.54279	0.42386
7	0.01840	0.14496	0.58118	0.43622	0.15391	0.59219	0.43828
8	0.02071	0.18496	0.63234	0.44738	0.19119	0.63946	0.44827
9	0.02288	0.22733	0.68126	0.45393	0.23058	0.68472	0.45414
10	0.02500	0.27196	0.72804	0.45609	0.27196	0.72804	0.45609

$n = 1000$

x	Najkrótszy γ_1^*	*lewy*	*prawy*	*dlugosc*	Clopper-Pearson *lewy*	*prawy*	*dlugosc*
50	0.01979	0.03678	0.06473	0.02795	0.03734	0.06539	0.02806
100	0.02165	0.08159	0.11972	0.03812	0.08211	0.12029	0.03818
150	0.02254	0.12797	0.17317	0.04519	0.12843	0.17366	0.04523
200	0.02312	0.17523	0.22574	0.05051	0.17562	0.22616	0.05054
250	0.02356	0.22310	0.27771	0.05460	0.22343	0.27805	0.05462
300	0.02391	0.27146	0.32919	0.05773	0.27172	0.32946	0.05774
350	0.02421	0.32022	0.38027	0.06005	0.32042	0.38047	0.06005
400	0.02449	0.36934	0.43099	0.06165	0.36947	0.43112	0.06165
450	0.02475	0.41879	0.48138	0.06259	0.41885	0.48144	0.06259
500	0.02500	0.46855	0.53145	0.06290	0.46855	0.53145	0.06290

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In[1]:= << Statistics'ContinuousDistributions'
 n =. ;
 x =. ;
 q =. ;
 Obet[a_,b_,q_]=Quantile[BetaDistribution[a,b],q];
 Lower[n_,x_,q_]=Obet[x,n-x+1,q];
 Upper[n_,x_,q_]=Obet[x+1,n-x,q];
 Leng[n_,x_,q_,r_]=Upper[n,x,q+r]-Lower[n,x,r];
In[2]:= n =20;(*input n*)
 x =7;(*input x between 2 and n-2*)
 q =0.95;(*input confidence level*)
 rr=r/.FindMinimum[Leng[n,x,q,r],{r,0,1-q}][[2]](*out gamma1*)
 Lower[n,x,rr](*out left end*)
 Upper[n,x,q+rr](*out right end*)
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Wybrana literatura

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