XXXV Konferencja Statystyka Matematyczna

Wisła -7-11/ XII 2009

Statistical Inference for Image Symmetries

Mirek Pawlak

pawlak@ee.umanitoba.ca

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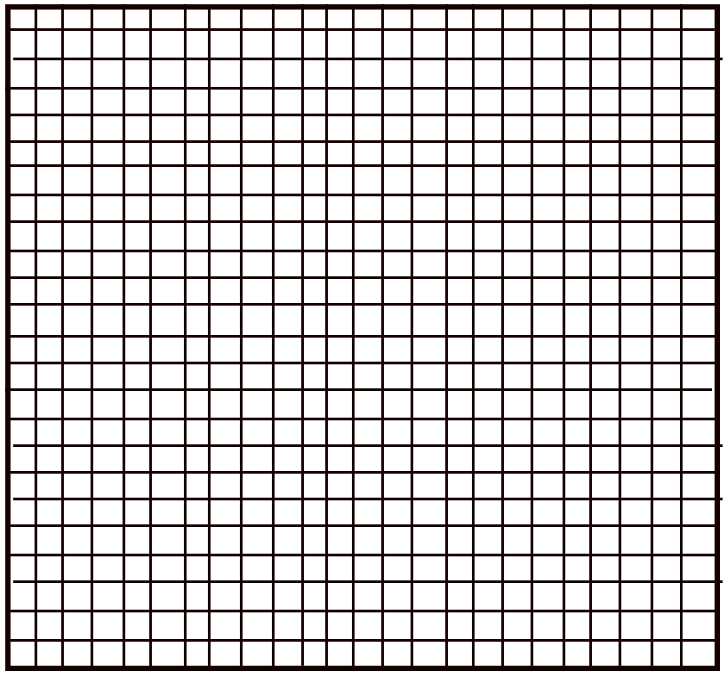
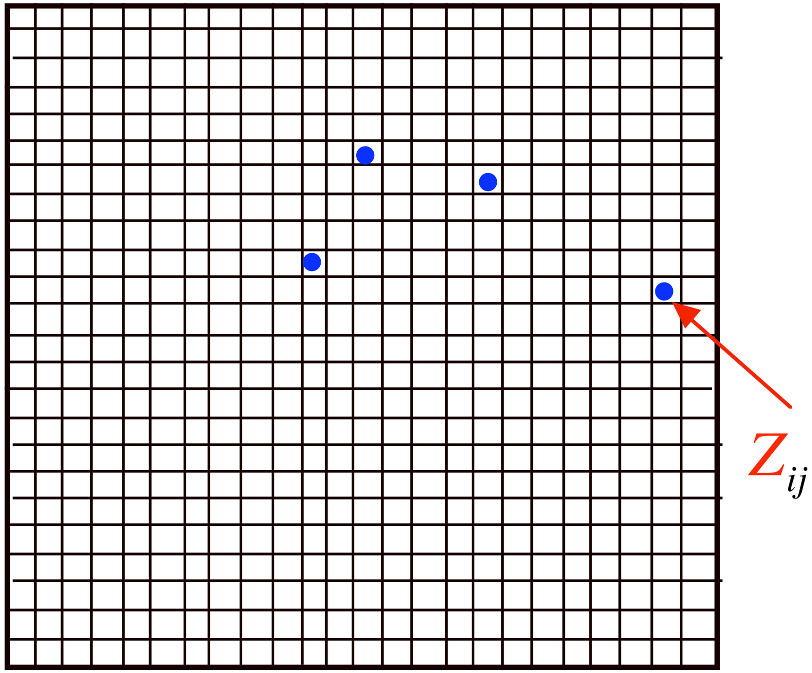
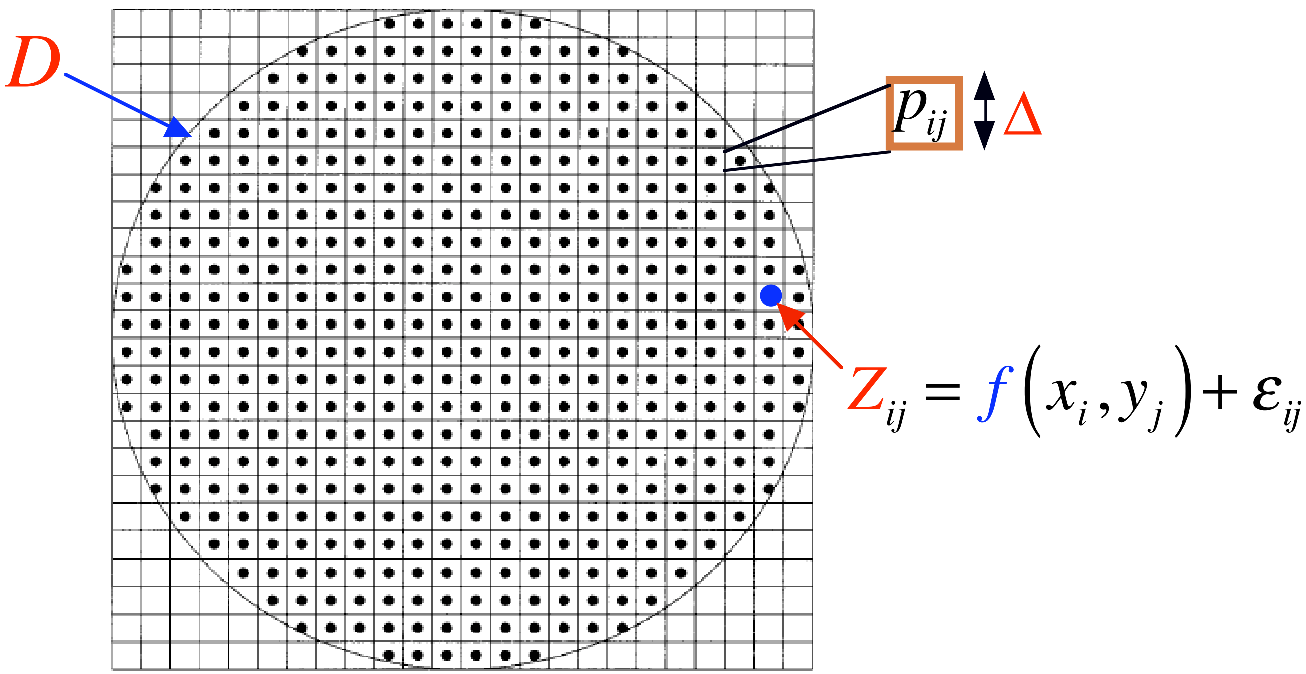
I Problem Statement





$$Z_{ij} = f\left(x_i, y_j\right) + \varepsilon_{ij}$$



# of data points $\propto n^2$; $\Delta \propto n^{-1}$

Problem 1: Let $f \in L_2(\mathbf{D})$. Given

$$Z_{ij} = f\left(x_i, y_j\right) + \varepsilon_{ij},\ 1 \le i, j \le n$$

and knowing that

$$f(x,y) = f\left(x\cos\left(2\beta^*\right) + y\sin\left(2\beta^*\right), x\sin\left(2\beta^*\right) - y\cos\left(2\beta^*\right)\right)$$

some $\beta^* \in [0,\pi)$, estimate β^*.

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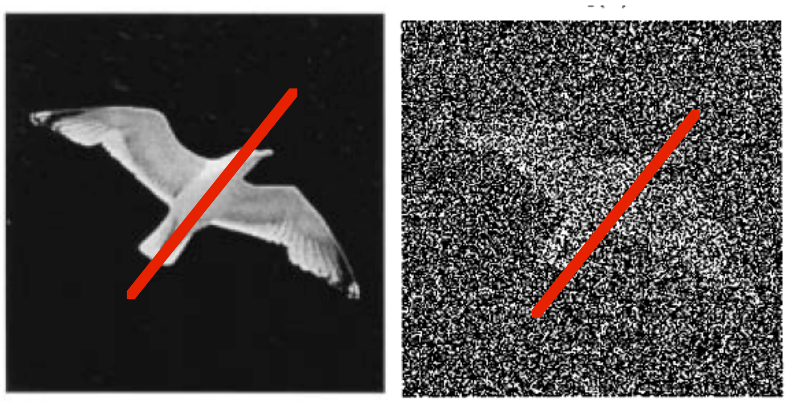


Problem 2: Let $f \in L_2(\mathbf{D})$. Given

$$Z_{ij} = f\left(x_i, y_j\right) + \varepsilon_{ij},\ 1 \le i, j \le n$$

verify whether the null hypothesis

$$H^S : f(x,y) \equiv (Sf)(x,y)$$

is true or not

$(Sf)(x,y)$: **Reflectional** Symmetry, **Rotational** Symmetry

$d=2$ $d=3$ $d=4$ $d=8$

An image becomes invariant under rotations through an angle $\frac{2\pi}{d}$

$$d=\infty$$

Radially Symmetric Objects

$$f(x,y)=g\left(\sqrt{x^2+y^2}\right)$$

➤ **Symmetry**

→ Hermann Weyl, *Symmetry*, Princenton Univ. Press, 1952.

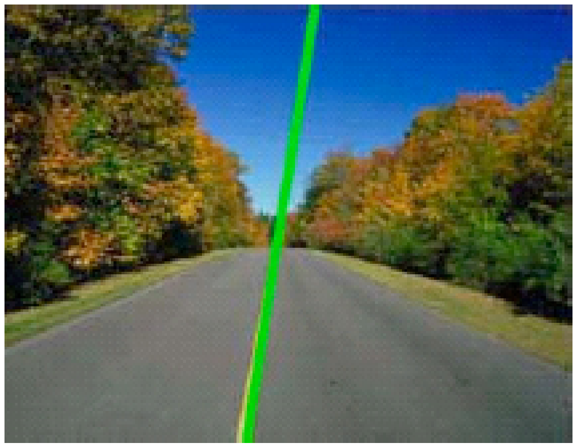
→J.Rosen, *Symmetry in Science: An Introduction to the General Theory*, Springer, 1995.

→ J.H. Conway, H. Burgiel, and C. Goodman-Strauss, *The Symmetry of Things*, A K Peters, 2008.

→ M. Livio, *The Equation That Couldn't Be Solved: How Mathematical Genius Discovered the Language of Symmetry*, Simon & Schuster, 2006.

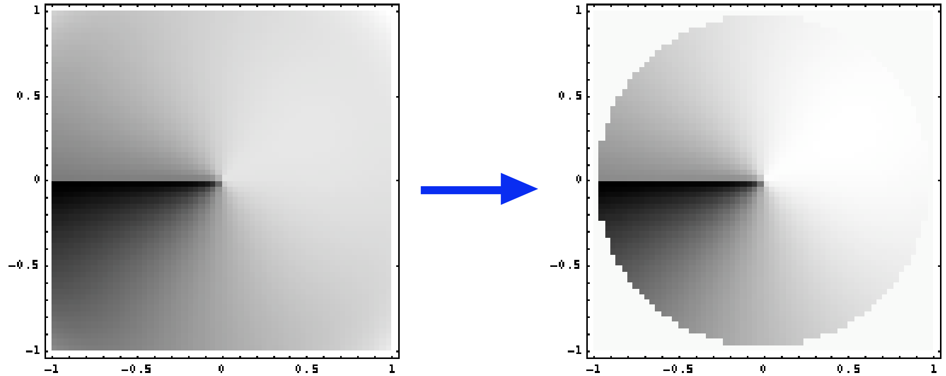
"...Livio writes passionately about the role of **symmetry** in human perception, arts,...."

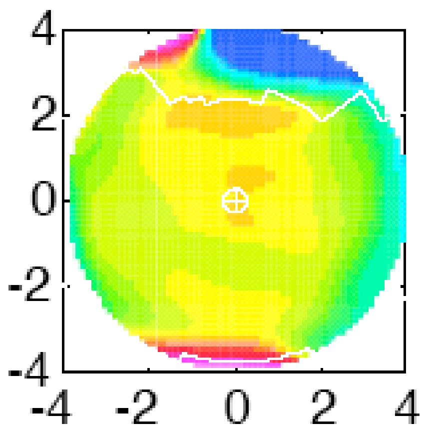

AE




YES WE CAN
NO WEEK END

II Image Representation in the Radial Basis Domain





Radial **Moments** and **Expansions (Zernike Basis)**

- $A_{pq}(f) = \iint_D f(x,y) V_{pq}^*(x,y)\,dxdy$

$$= \int_0^{2\pi}\int_0^1 f(\rho,\theta) R_{pq}(\rho)\, e^{-iq\theta} \rho\, d\rho\, d\theta$$

Degree

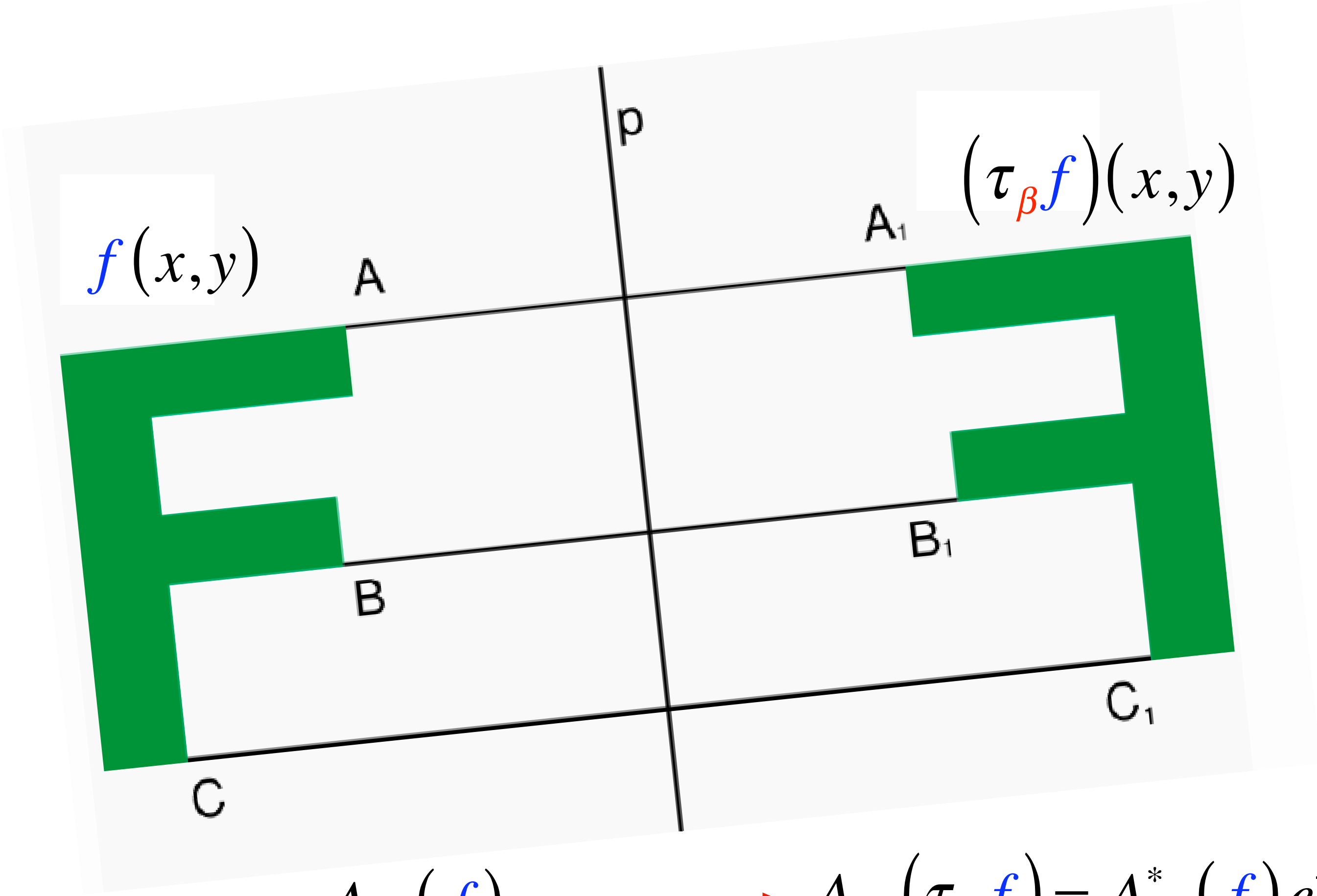
Angular dependence

Bhatia & Born: "On circle polynomials of Zernike and related orthogonal sets". **Proc. Cambr. Phil. Soc**. 1954

- $L_2(D) \ni f(x,y) \simeq \sum_{p=0}^{\infty} \sum_{|q| \leq p} \frac{p+1}{\pi} A_{pq}(f) V_{pq}(x,y)$

Invariant Properties

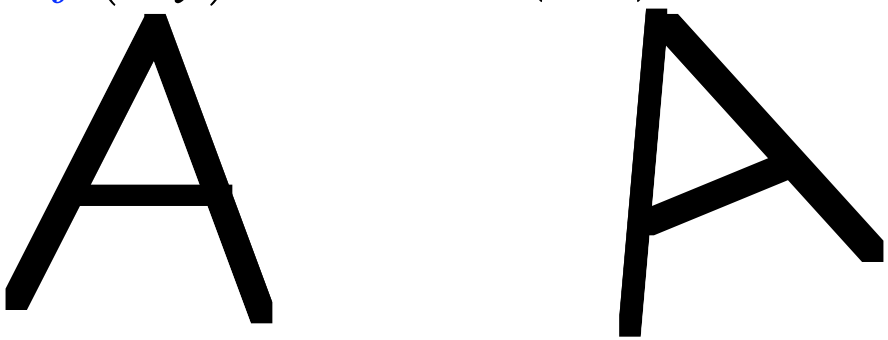
→ Reflection



$$A_{pq}(f) \longrightarrow A_{pq}(\tau_\beta f) = A^*_{pq}(f)\, e^{-i2q\beta}$$

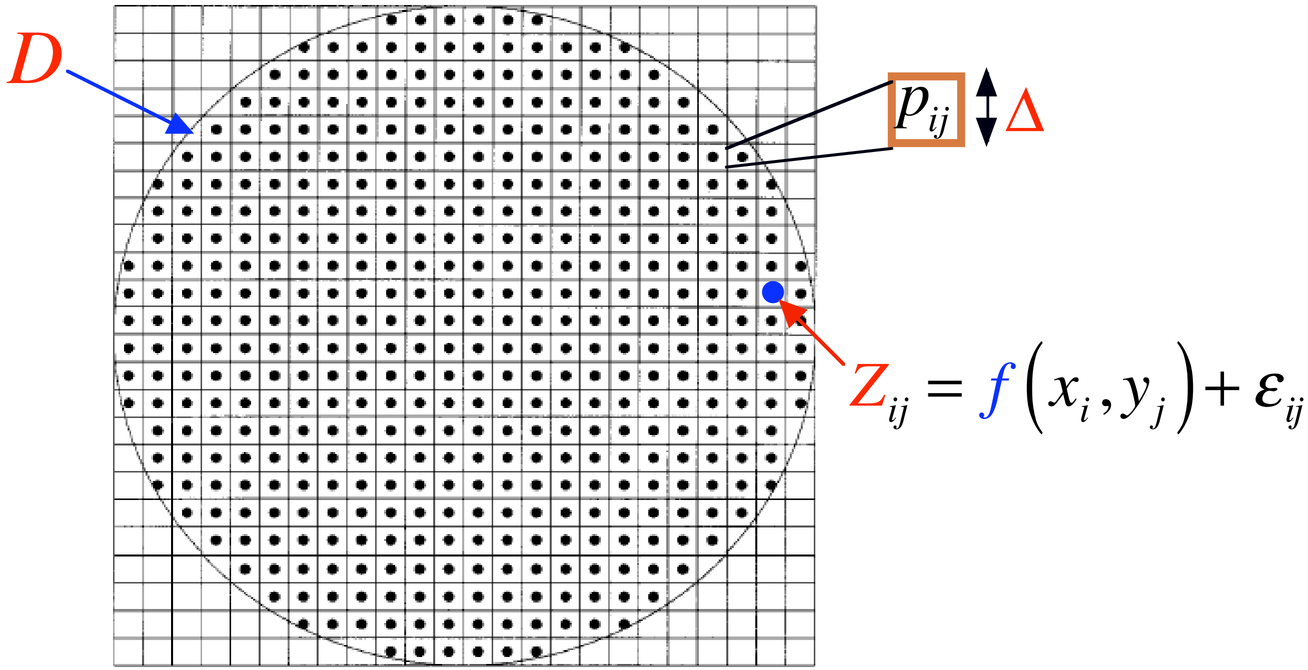
→ Rotation

$f(x,y)$ $\qquad$ $(r_\alpha f)(x,y)$



$$A_{pq}(f) \longrightarrow A_{pq}(r_\alpha f) = A_{pq}(f)\, e^{-iq\alpha}$$

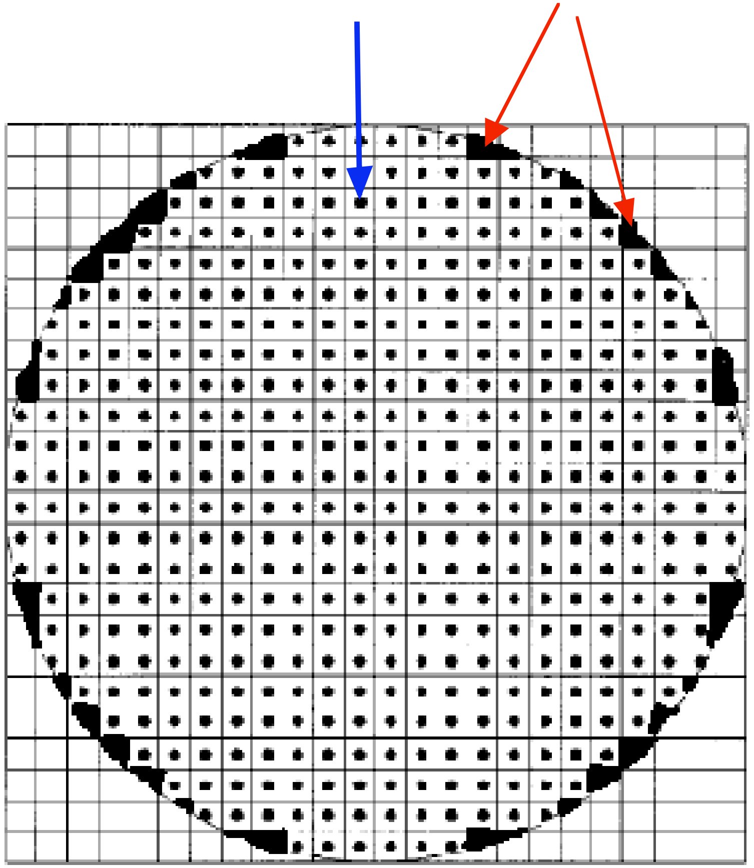
$A_{pq}(f)$ from noisy data



$$\hat{A}_{pq}(f) = \sum_{(x_i, y_j) \in D} w_{pq}(x_i, y_j) Z_{ij}$$

$$w_{pq}(x_i, y_j) = \iint_{p_{ij}} V_{pq}^*(x, y)\,dxdy \approx \Delta^2 V_{pq}^*(x_i, y_j)$$

$$E\hat{A}_{pq} = A_{pq}(f) + D_{pq}(\Delta) + G_{pq}(\Delta)$$



$$D_{pq}(\Delta) = O(\Delta)$$

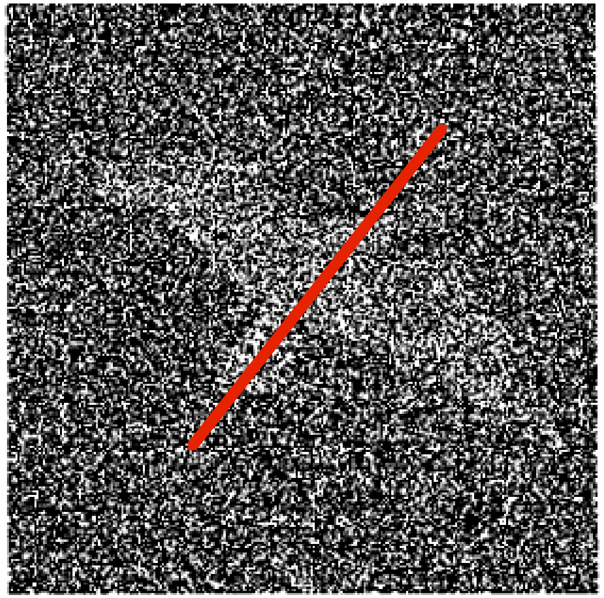
$$G_{pq}(\Delta) = O(\Delta^{\gamma})$$

$$1 < \gamma = \frac{285}{208}$$

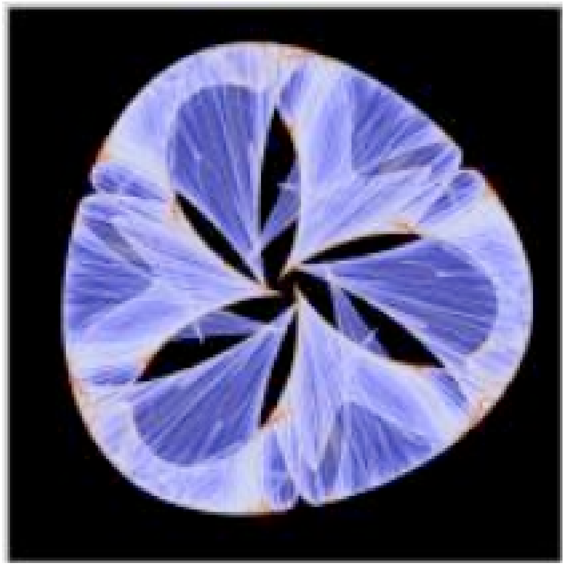
*Gauss lattice points problem of a circle *

III Semiparametric Inference for Image Symmetry

Estimation of symmetry parameters: the **axis of symmetry** (β), the **degree** of rotational symmetry (d) of **nonparametric** image function: $\theta = \theta(f), f \in L_2(D)$.



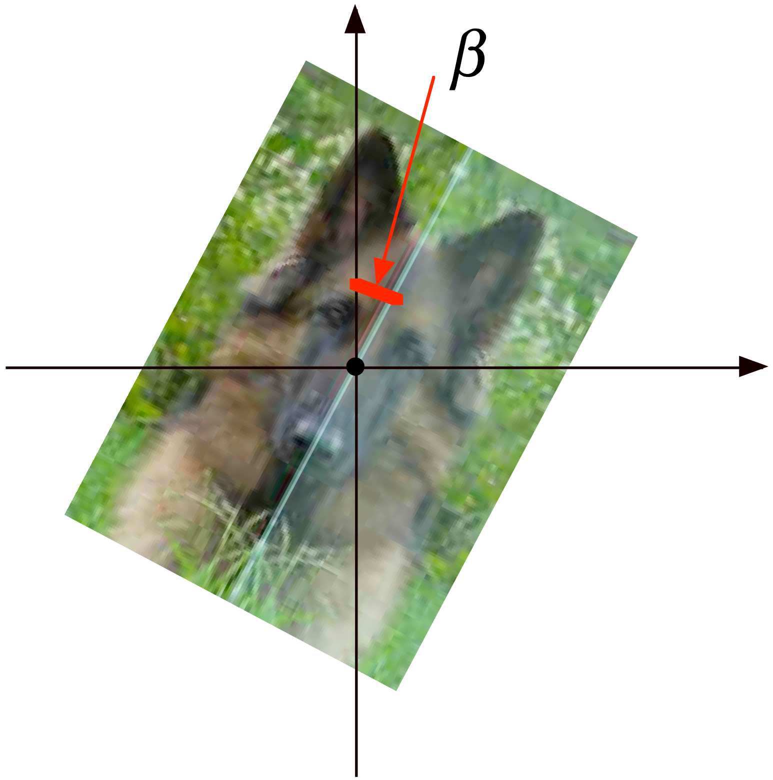
β^*



d^*

Semiparametric Inference on the Axis of Reflectional Symmetry

$$A_{pq}\left(\tau_\beta f\right) = e^{-2iq\beta} A_{p,-q}\left(f\right)$$



- **Contrast Function**

$$M_N(\beta, f) = \sum_{p=0}^{N} \frac{p+1}{\pi} \sum_{q=-p}^{p} \left| A_{pq}(f) - e^{-2iq\beta} A_{p,-q}(f) \right|^2$$

$$\xrightarrow{N}$$

$$M(\beta, f) = \sum_{p=0}^{\infty} \frac{p+1}{\pi} \sum_{q=-p}^{p} \left| A_{pq}(f) - e^{-2iq\beta} A_{p,-q}(f) \right|^2$$

$$= \left\| f - \tau_\beta f \right\|^2$$

- **Assumption**

Suppose that $f \in L_2(D)$ is **invariant** under some unique reflection τ_{β^*}

Hence, if $\tau_{\beta^*} f = f$ then we have

$$M(\beta^*, f) = 0$$

or

$$M_N(\beta^*, f) = 0 \text{ for all } N$$

- **Uniqueness** of β^*

Lemma U: Under **Assumption** the solution of

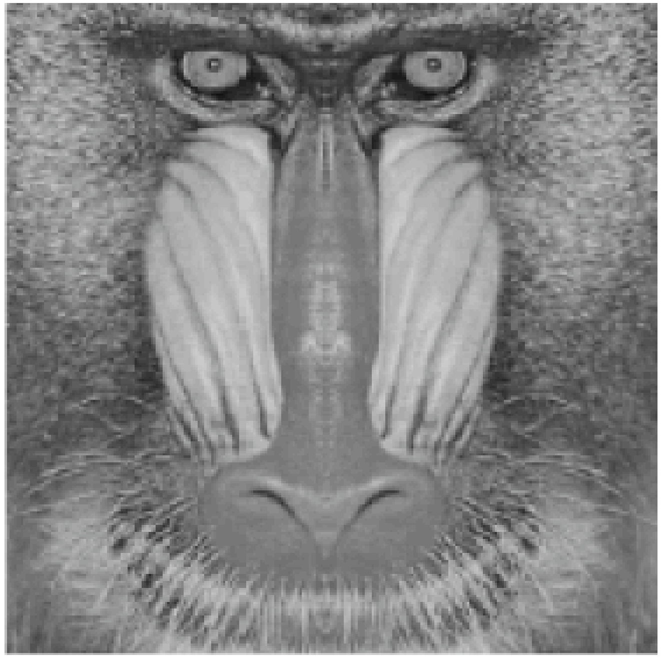
$$M_N(\beta, f) = 0$$

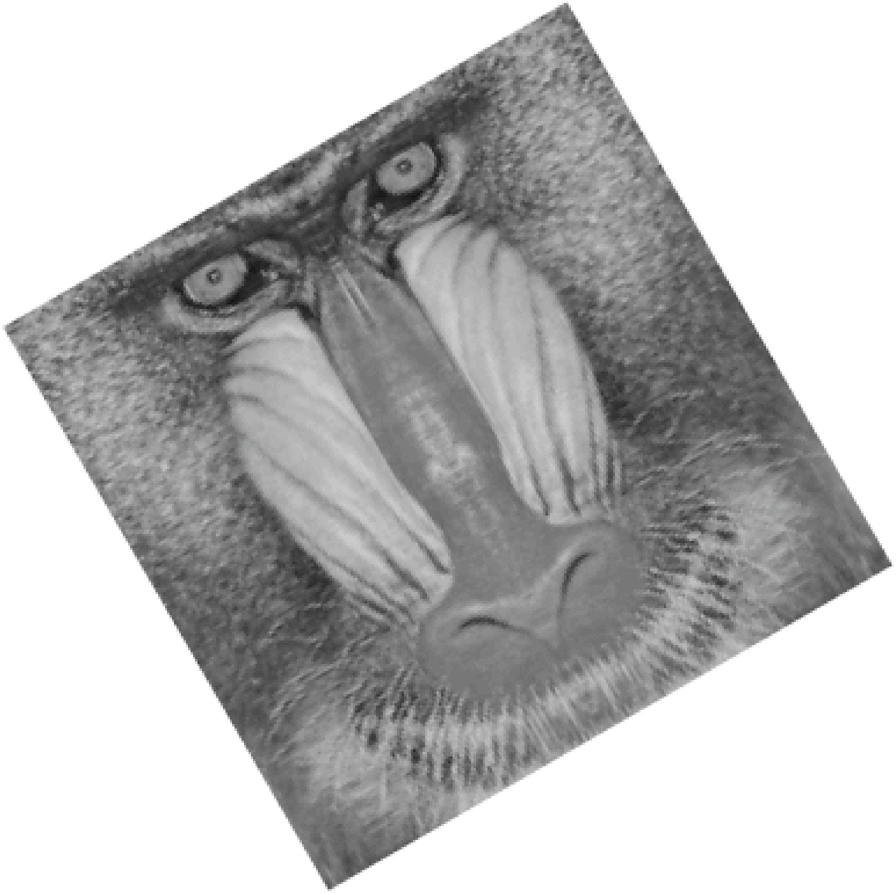
is uniquely determined and must equal β^* if we choose N so large that $M_N(\beta, f)$ contains $\left\{A_{pq}(f) \neq 0\right\}$ for which the **gcd** of q's is 1.

$$\left\{A_{p_1 q_1}(f) \neq 0, \ldots, A_{p_r q_r}(f) \neq 0\right\},\ p_i \leq N, i = 1, .., r$$

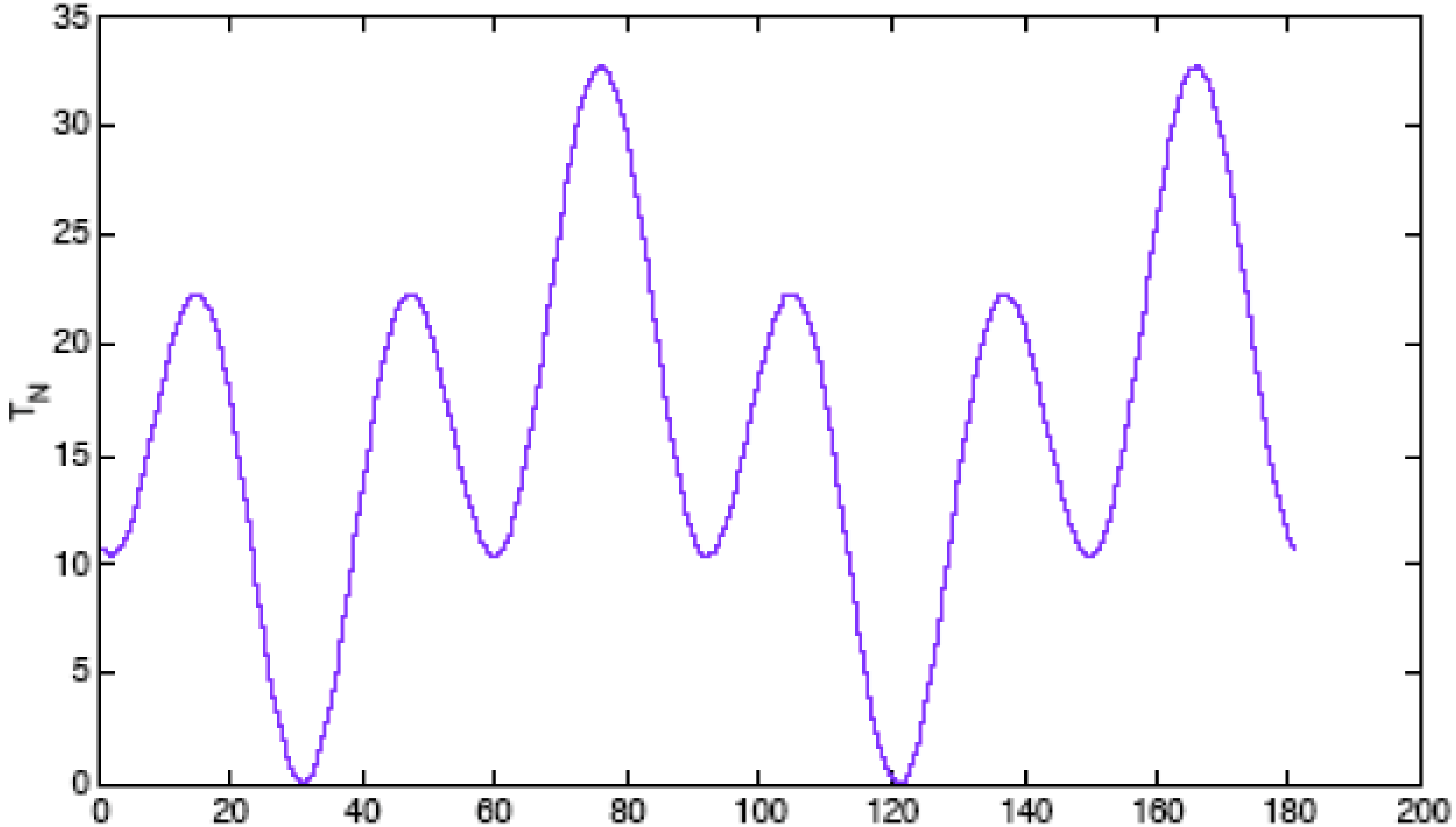
$$\mathrm{gcd}(q_1, \ldots, q_r) = 1.$$

$\beta^* = 120^\circ$

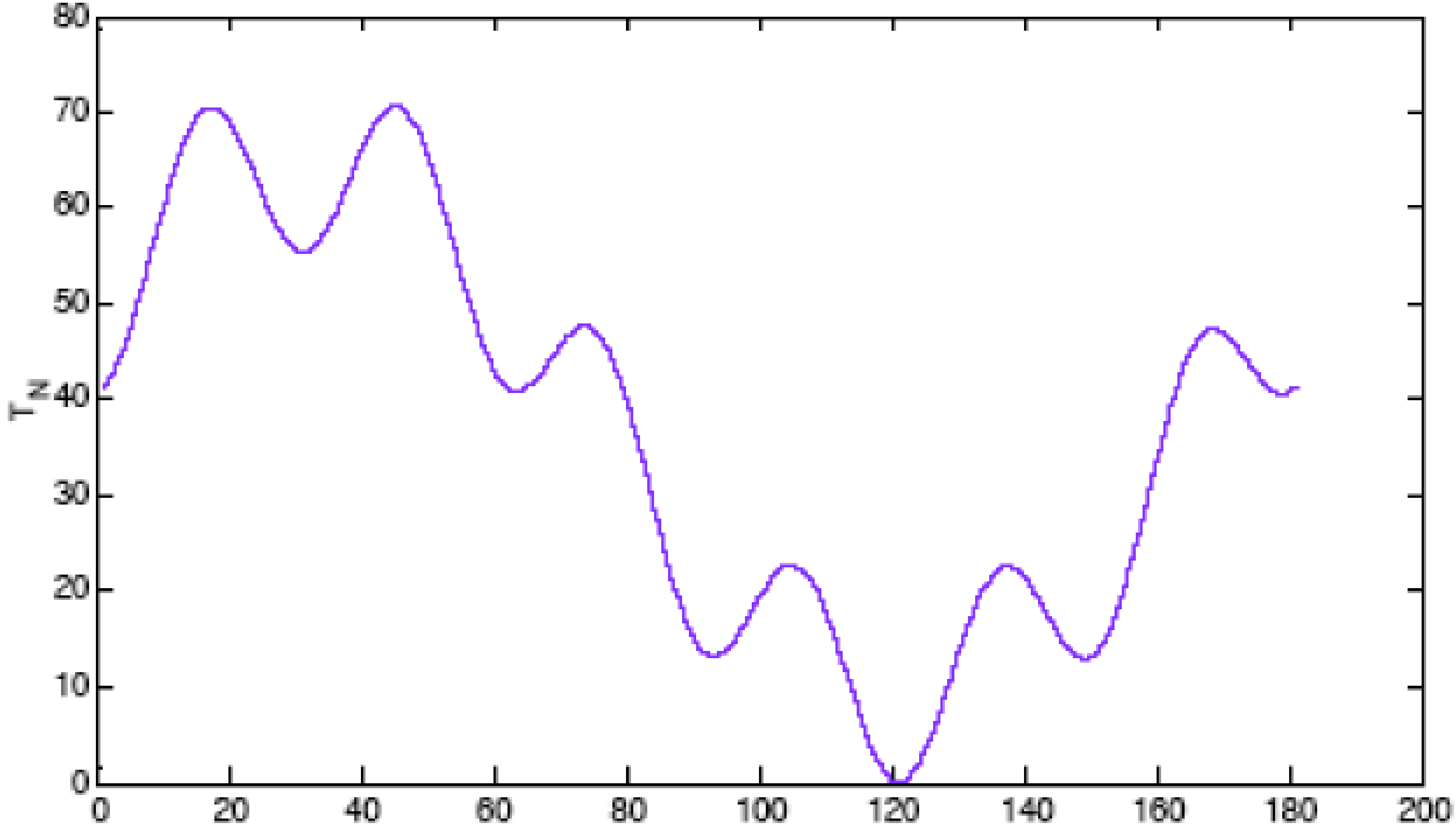




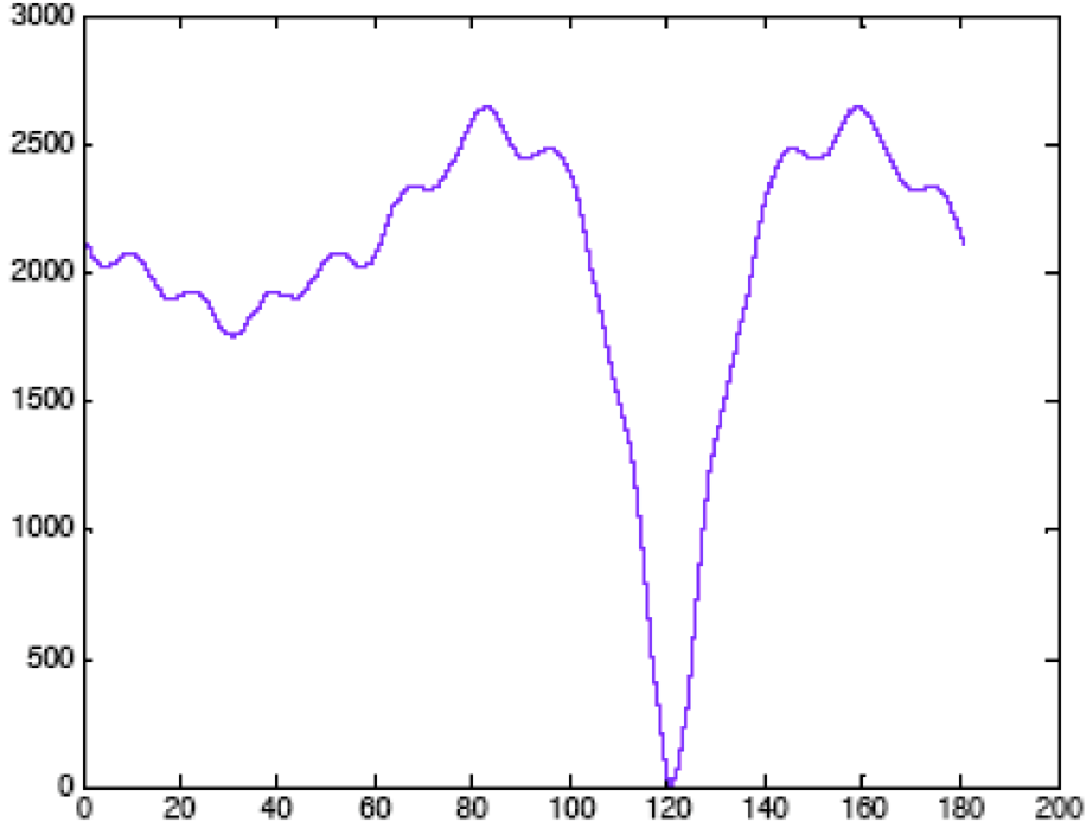
$M_N(\beta, f)$ based only on $A_{42}(f), A_{86}(f)$



$M_N(\beta, f)$ based only on $A_{51}(f), A_{86}(f)$



$$M_{14}(\beta, f) = \sum_{p=0}^{14} \frac{p+1}{\pi} \sum_{q=-p}^{p} \left| A_{pq}(f) - e^{-2iq\beta} A_{p,-q}(f) \right|^2$$



$$A_{pq}(f) = \left|A_{pq}(f)\right| e^{i r_{pq}(f)}$$

$$\Rightarrow \quad M_N(\beta, f) = \sum_{p=0}^{N} \frac{p+1}{\pi} \sum_{q=-p}^{p} 4\left|A_{pq}(f)\right|^2 \left(1 - \cos\left(2r_{pq}(f) + 2q\beta\right)\right)$$

$\Rightarrow$

β^* :

$$r_{pq}(f) + q\beta^* = l\pi$$

for all (p,q) with $\left|A_{pq}(f)\right| \neq 0$.

• Estimated Contrast Function

$$\hat{M}_N(\beta) = \sum_{p=0}^{N} \frac{p+1}{\pi} \sum_{q=-p}^{p} \left| \hat{A}_{pq} - e^{-2iq\beta} \hat{A}_{p,-q} \right|^2$$

$$= \sum_{p=0}^{N} \frac{p+1}{\pi} \sum_{q=-p}^{p} 4 \left| \hat{A}_{pq} \right|^2 \left(1 - \cos\left(2\hat{r}_{pq} + 2q\beta \right) \right)$$

$$\hat{\beta}_{\Delta,N} = \arg\min_{\beta \in [0,\pi)} \hat{M}_N(\beta)$$

→ van der Vaart: **Asymptotic Statistics**, 1998.

• Accuracy of $\hat{\beta}_{\Delta,N}$

Theorem 3.1

Let $f \in BV(D)$ and be reflection invariant w.r.t. a unique axis $\beta^* \in [0,\pi)$. Let N be so large that β^* is determined as the unique zero of $M_N(\beta, f)$. Then

$$\hat{\beta}_{\Delta,N} \to \beta^* \text{ (P) as } \Delta \to 0$$

Proof: $$\sup_{\beta\in[0,\pi)} \left| \hat{M}_N(\beta) - M_N(\beta, f) \right| \to 0 \text{ (P)}$$

Theorem 3.2

$$\hat{\beta}_{\Delta,N} = \beta^* + O_P(\Delta) \quad \left(\rightarrow \sqrt{n^2} \text{ rate} \leftarrow\right)$$

Proof:

- $$0 = \hat{M}_N^{(1)}\left(\hat{\beta}_{\Delta,N}\right) = \hat{M}_N^{(1)}\left(\beta^*\right) + \hat{M}_N^{(2)}\left(\tilde{\beta}\right)\left(\hat{\beta}_{\Delta,N} - \beta^*\right)$$
- $$\hat{\beta}_{\Delta,N} - \beta^* = -\frac{\hat{M}_N^{(1)}\left(\beta^*\right)}{\hat{M}_N^{(2)}\left(\tilde{\beta}\right)}$$

- From **Theorem 3.1**:

$$\sup_{\beta\in[0,\pi)} \left| \hat{M}_N^{(2)}(\beta) - M_N^{(2)}(\beta, f) \right| \to 0 \ \ \text{(P)}$$

$$\tilde{\beta} \to \beta^* \ \ \text{(P)}$$

$$\hat{M}_N^{(2)}\left(\tilde{\beta}\right) \to M_N^{(2)}\left(\beta^*, f\right)$$

$$= \sum_{p=0}^{N} \frac{p+1}{\pi} \sum_{q=-p}^{p} 16q^2 \left| A_{pq}\left(f\right) \right|^2 \neq 0$$

- $\Delta^{-1}\left(\hat{\beta}_{\Delta,N} - \beta^{*}\right) \approx -\dfrac{\Delta^{-1}\hat{M}_{N}^{(1)}\left(\beta^{*}\right)}{M_{N}^{(2)}\left(\beta^{*}, f\right)}$

- $\hat{M}_{N}^{(1)}\left(\beta^{*}\right) = \sum_{p=0}^{N} \dfrac{p+1}{\pi} \sum_{q=-p}^{p} 8q\left|\hat{A}_{pq}\right|^{2} \sin\left(2\hat{r}_{pq} + 2q\beta^{*}\right)$

$$\left(\rightarrow r_{pq}(f) + q\beta^{*} = l\pi \leftarrow\right)$$

$$= \sum_{p=0}^{N} \frac{p+1}{\pi} \sum_{q=-p}^{p} 8q\left|\hat{A}_{pq}\right|^{2} \sin\left(2\left(\hat{r}_{pq} - r_{pq}(f)\right)\right)$$

$$\approx \sum_{p=0}^{N} \frac{p+1}{\pi} \sum_{q=-p}^{p} 8q \left| A_{pq}(f) \right|^2 \sin\left(2\left(\hat{r}_{pq} - r_{pq}(f)\right)\right)$$

- $\hat{A}_{pq} = A_{pq}(f) + O_P(\Delta)$
- $\hat{r}_{pq} - r_{pq}(f) = O_P(\Delta)$ ■

Theorem 3.3 Let f be Lip(1). $E\varepsilon^4 < \infty$.

$$\Delta^{-1}\left(\hat{\beta}_{\Delta,N} - \beta^*\right) \Rightarrow N\left(0, \frac{8\sigma^2}{M_N^{(2)}\left(\beta^*, f\right)}\right)$$

$$M_N^{(2)}\left(\beta^*, f\right) = \sum_{p=0}^{N} \frac{p+1}{\pi} \sum_{q=-p}^{p} 16q^2 \left|A_{pq}(f)\right|^2$$

⇒ **Confidence Interval** for β^*

$$\left[\hat{\beta}_{\Delta,N} - \Phi^{-1}(1-\alpha)\frac{2\sqrt{2}\Delta\hat{\sigma}}{\sqrt{\hat{M}_N^{(2)}}}, \hat{\beta}_{\Delta,N} + \Phi^{-1}(1-\alpha)\frac{2\sqrt{2}\Delta\hat{\sigma}}{\sqrt{\hat{M}_N^{(2)}}}\right]$$

- $\hat{M}_N^{(2)} = \sum_{p=0}^{N} \frac{p+1}{\pi} \sum_{q=-p}^{p} 16q^2 \left|\hat{A}_{pq}\right|^2$
- $\hat{\sigma}^2 = \frac{1}{4C(\Delta)} \sum_{(x_i,y_j),(x_{i+1},y_j),(x_i,y_{j+1})\in D} \left\{\left(Z_{i,j} - Z_{i+1,j}\right)^2 + \left(Z_{i,j} - Z_{i,j+1}\right)^2\right\}$

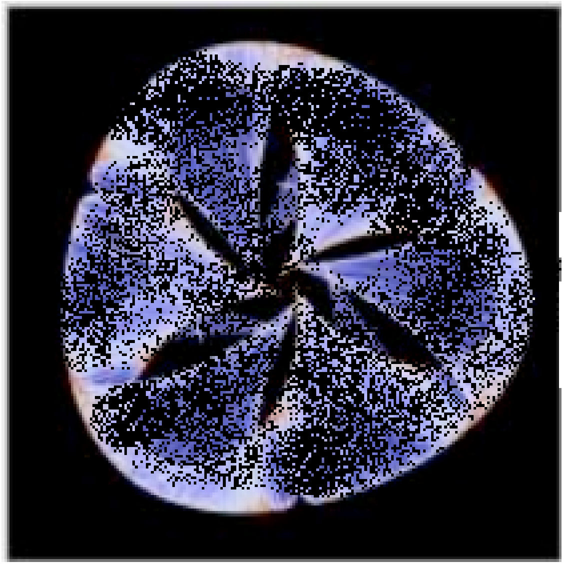
Remark 1: f is not reflection symmetric

$$\hat{\beta}_{\Delta,N} \to \beta^{\circ} = \arg\min_{\beta} \left\| f - \tau_{\beta} f \right\|^2$$

Original

"symmetrized"

Remark 2: Estimation d $\left(\in\{2,3,4,....\}\right)$ for images invariant under rotations through an angle $\frac{2\pi}{d}$?



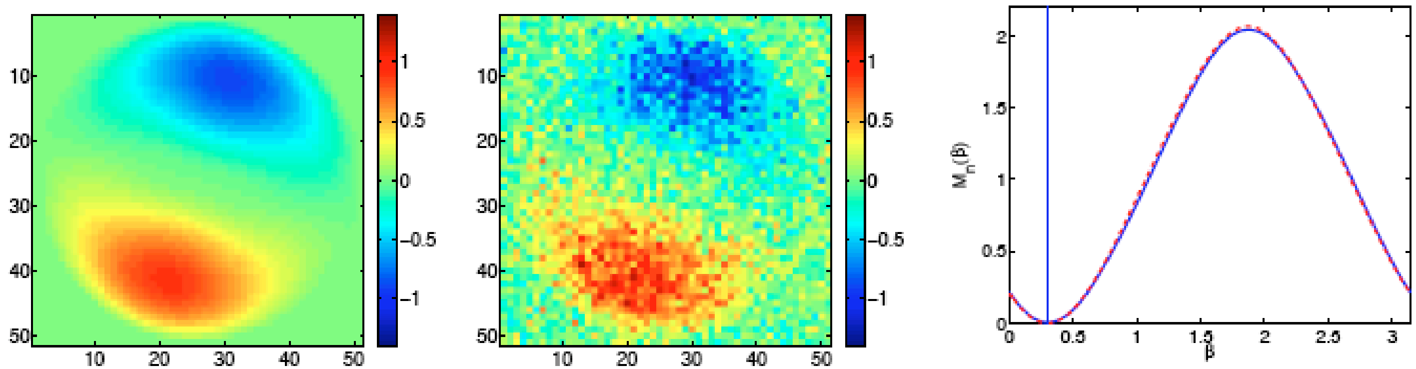


Figure 1: Reflection symmetric function f_1 without noise, with noise, and $M_7(\beta)$ (full curve) and $\hat{M}_7(\beta)$ (dashed curve). Parameters are $n = 25$ and signal-to-noise-ratio$= 5$. The vertical line indicates the direction of reflection symmetry in the true image.

$$f_1(\rho, \theta) \;=\; c_1 \cdot x \cdot (1-\rho) \cdot \left(\sin(y + \sqrt{x^2 + y^4}) + \sin(-y + \sqrt{x^2 + y^4}) \right)$$

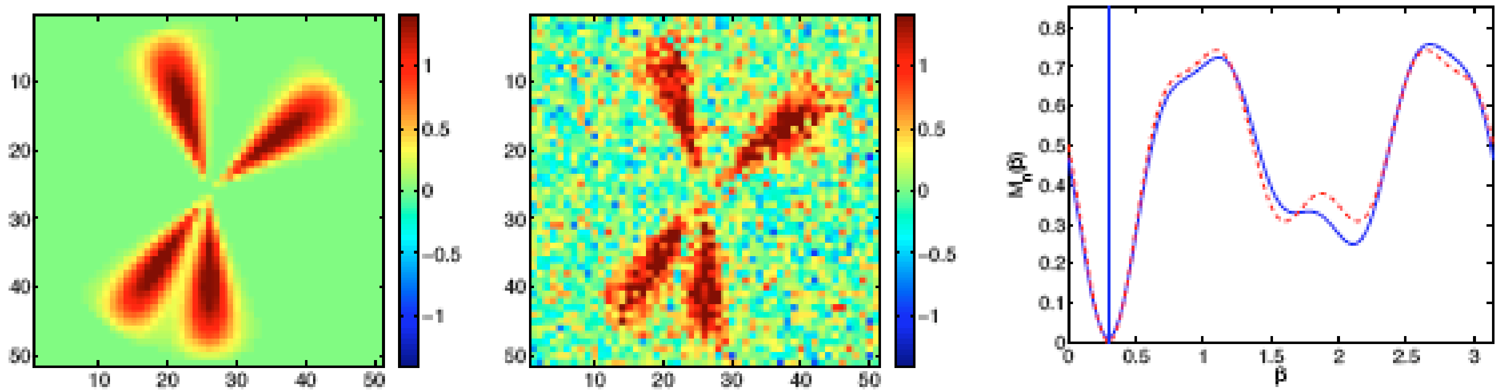


Figure 2: Reflection symmetric function f_2 without noise, with noise, and $M_7(\beta)$ (full curve) and $\hat{M}_7(\beta)$ (dashed curve). Parameters are $n = 25$ and signal-to-noise-ratio$= 5$. The vertical line indicates the direction of reflection symmetry in the true image.

$$
\begin{aligned}
f_2(\rho, \theta) \;&=\; c_2 \cdot \rho \cdot (1-\rho) \cdot \big(e^{\cos(\theta)/0.02} + e^{\cos(\theta+0.6)/0.02} + e^{\cos(\theta-0.3+\pi)/0.02} \\
&\quad + e^{\cos(\theta+0.9+\pi)/0.02} \big) \,,
\end{aligned}
$$

IV Testing for Image Symmetries

Testing Rotational Symmetries

$d = 2$ - testing image symmetry w.r.t. rotation through π

$$A_{pq}\left(f\right) \rightarrow A_{pq}\left(r_d f\right) = e^{2\pi iq/d} A_{pq}\left(f\right) = e^{\pi iq} A_{pq}\left(f\right)$$

The **orthogonal projection**

$$\left\| f - r_d f \right\|^2 = \sum_{p=0}^{\infty} \sum_{|q| \leq p} \frac{p+1}{\pi} \left| 1 - e^{2\pi i q/d} \right|^2 \left| A_{pq}(f) \right|^2$$

The **test statistic (d=2)**

$$T_N^{r_2} = \sum_{\substack{p=0 \\ p=odd}}^{N} \sum_{|q| \leq p} \frac{p+1}{\pi} \left| \hat{A}_{pq} \right|^2$$

Theorem 4.1:

A: Under $H^{r_2}: r_2 f = f$ let

$$N \to \infty,\; N^7 \Delta \to 0, \text{ as } \Delta \to 0$$

Then for

$$T_N^{r_2} = \sum_{\substack{p=0 \\ p=odd}}^{N} \sum_{|q| \le p} \frac{p+1}{\pi} \left| \hat{A}_{pq} \right|^2$$

we have

$$\frac{T_N^{r_2} - \sigma^2 \Delta^2 a(N)}{\Delta^2 \sqrt{a(N)}} \Rightarrow N\left(0, 2\sigma^4\right) \qquad a(N) = \begin{cases} N(N+2)/4 - & N \text{ even} \\ (N+1)(N+3)/4 - N & \text{odd} \end{cases}$$

B: Under $\bar{H}^{r_2}: r_2 f \neq f$ let $f \in C^s(D), s \geq 2$ and let

$$N \to \infty,\ N^{3/2}\Delta^{\gamma-1} \to 0,\ N^{2s+1}\Delta \to \infty \text{ as } \Delta \to 0$$

Then we have

$$\frac{T_N^{r_2} - \|f - r_2 f\|^2 / 4}{\Delta} \Rightarrow N\left(0, \sigma^2 \|f - r_2 f\|^2\right)$$

Proof:

- Under $H^{r_2}: T_N^{r_2}$ is a **quadratic form** of iid random variables

(*de Jong: A CLT for generalized quadratic forms, *Pr.Th.Related Fields*, 1987*)

- Under $\bar{H}^{r_2}: T_N^{r_2}$ is dominated by a **linear term**

Similar results exist for

- $d=4$ - testing image symmetry w.r.t. rotation by $\pi/2$
- $d=\infty$ - testing image radiality, i.e., $f(x,y)=g(\rho)$
- Testing image **mirror symmetry**
- Joint hypothesis

$$H^{\tau_y,r_2}: r_2 f = f \text{ AND } \tau_y f = f$$

$$\bar{H}^{\tau_y,r_2}: r_2 f \neq f \text{ OR } \tau_y f \neq f$$

V References

Symmetry

→ Hermann Weyl, *Symmetry*, Princenton Univ. Press, 1952.

→J.Rosen, *Symmetry in Science: An Introduction to the General Theory*, Springer, 1995.

→ J.H. Conway, H. Burgiel, and C. Goodman-Strauss, *The Symmetry of Things*, A K Peters, 2008.

→ M. Livio, *The Equation That Couldn't Be Solved: How Mathematical Genius Discovered the Language of Symmetry*, Simon & Schuster, 2006.

"...Livio writes passionately about the role of **symmetry** in human perception, arts,...."

Symmetry & Statistics

- Given iid (p) sequence $Z_1,\ldots,Z_n$ verify whether

$$H_0 : p(z) = p(-z)$$

- $$Y_{i,n} = m(x_{i,n}) + Z_{i,n}$$

$$H_0 : p(z) = p(-z)$$

$\rightarrow$ Fan & Gencay(95), Ahmad & Li (97): $m(\bullet)$ - linear regression

$\rightarrow$ Dette et al. (02) : $m(\bullet)$ - nonparametric regression

- Bissantz, Holzmann, and Pawlak, Testing for image symmetries -- with application to confocal microscopy, **IEEE Trans. Inform. Theory**, 2009.
- Bissantz, Holzmann, and Pawlak, Estimating the axis of reflectional symmetry of an image, **Annals of Applied Statistics**, to appear.
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