Resampling for nonstationary stochastic models

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Plan of the presentation

* Nonstationary stochastic models
  - PC and APC models, time domain
  - PC and APC frequency domain
  - periodic counting process models
* Limit results
* Why resampling?
* Selected results
* Future directions of research

Time domain approach

$\{X_t : t \in \mathbb{Z}\}$ - APC, when

$$\mu_X(t) = E(X_t)$$

and the autocovariance function

$$B_X(t,\tau) = \text{cov}(X_t, X_{t+\tau})$$

are almost periodic function at t for every $\tau \in \mathbb{Z}$.
Put $B_X(t,\tau) = \sum_{\lambda\in\Lambda} a(\lambda,\tau)e^{i\lambda\tau}$
Time domain approach ($\mu_X(t) \equiv 0$):

$$\hat{a}_n(\lambda,\tau) = \frac{1}{n-\tau}\sum_{t=1}^{n-\tau} X(t+\tau)X(t)e^{-i\lambda t}$$

Frequency domain approach

Harmonizable time series $\{X(t) : t \in \mathbb{Z}\}$

$$X(t) = \int_0^{2\pi} e^{i\xi t} Z(d\xi).$$

Spectral bimeasure is defined as

$$R((a,b] \times (c,d]) = E[(Z(b) - Z(a))(Z(d) - Z(c))],$$

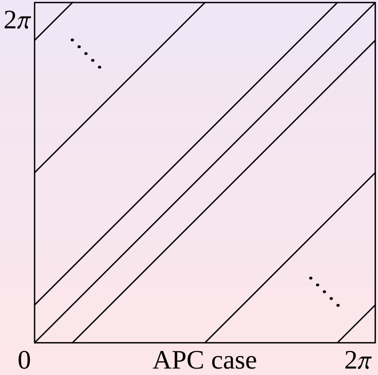
with a support

$$S = \bigcup_{\lambda \in \Lambda} \{(\xi_1, \xi_2) \in (0, 2\pi]^2 : \xi_2 = \xi_1 \pm \lambda\}.$$

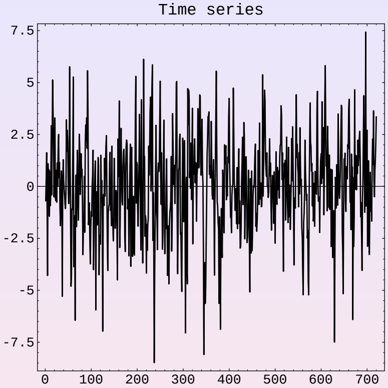
Spectral density estimator

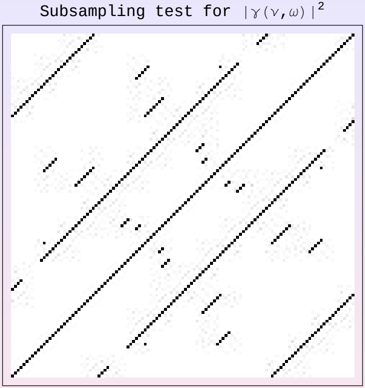
$$\hat{G}_n(\nu, \omega) = \frac{1}{2\pi n} \sum_{t=1}^{n} \sum_{s=1}^{n} K_n(s-t) X_t X_s e^{-i\nu t} e^{i\omega s}. \tag{1}$$

Support lines



Simulation example





$$X_t = (2 + \sin(2\pi t/4))Y_{t-1} + Y_t,$$
where Y_t are i.i.d. from $N(0,1)$.

Nonstationary counting process

X - counting process on $[0, T]$.
Intensity of X is of the form

$$\lambda(t) = \lambda_0(t) Y(t) \qquad , \qquad t \in [0, T]$$

- $\lambda_0(t)$ – nonnegative deterministic periodic function
- $Y(t)$ – nonnegative stochastic process

Nonstationary counting process

Sieve estimator of $\lambda_0(t)$

The histogram maximum likelihood estimator of the periodic $\lambda_0(t)$ function is of the form:

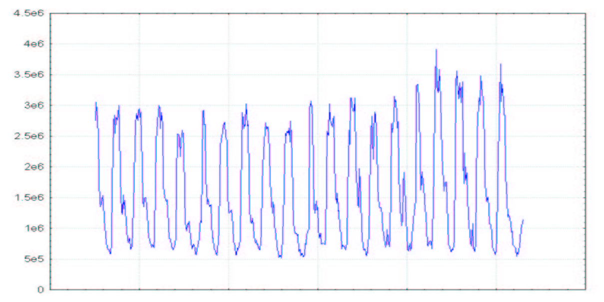
$$\widehat{\lambda}_n(s) = \frac{\sum_{k=1}^{n} X_k(B_n^s)}{\sum_{k=1}^{n} \int_{B_n^s} Y_k(u)du} \mathbf{1}_{D_n}(s), \quad s \in [0, P],$$

where $s \in B_n^s$ is the interval of the length P/b that contains s and

$$D_n = \{\sum_{k=1}^{n} \int_{B_n^s} Y_k(u)du > 0\}.$$

Real data example

Incoming packets number in one hour non-overlapping bins - border between the network of University of Waikato and the internet provider.



Limit results

Asymptotic normality, PC time domain domain

We have

$$\sqrt{n}\,(\hat{a}_n(\lambda,\tau) - a(\lambda,\tau)) \xrightarrow{d} \mathcal{N}_2(0,\Sigma), \qquad (2)$$

where

$$\Sigma = \begin{bmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{21} & \sigma_{22} \end{bmatrix},$$

$$\sigma_{11} = \frac{1}{T}\sum_{s=1}^{T}\sum_{k=-\infty}^{\infty} B_{Z_\tau}(s,k)\cos(\lambda s)\cos(\lambda k),$$

$$\sigma_{22} = \frac{1}{T}\sum_{s=1}^{T}\sum_{k=-\infty}^{\infty} B_{Z_\tau}(s,k)\sin(\lambda s)\sin(\lambda k),$$

$$\sigma_{12} = \sigma_{21} = \frac{1}{T}\sum_{s=1}^{T}\sum_{k=-\infty}^{\infty} B_{Z_\tau}(s,k)\cos(\lambda s)\sin(\lambda k),$$

and $Z(t,\tau) = X(t)X(t+\tau) - B_X(t,\tau)$,
$B_{Z_\tau}(s,k) = Cov\,(Z(s,\tau), Z(s+k,\tau))$.

Asymptotic covariance, APC case

Lemma (Lenart 2008)

If

(i) there exists $\delta > 0$ such that $\sup_{t\in\mathbb{Z}} \|X_t\|_{6+3\delta} \leq \Delta < \infty$,

(ii) $\sum_{k=1}^{\infty} k^2 \alpha(k)^{\frac{\delta}{2+\delta}} \leq K < \infty$,

(iii) $K_n(s-t) = I\{|s-t| \leq w_n\}$ + additional regularity assumptions

then we have a convergence

$$\lim_{n\to\infty} \frac{n}{w_n} cov\left(\hat{G}_n(\nu_1,\omega_1), \hat{G}_n(\nu_2,\omega_2)\right) = P(\nu_1,\nu_2)\overline{P(\omega_1,\omega_2)} + P(\nu_1, 2\pi-\omega_2)\overline{P(\nu_2, 2\pi-\omega_1)},$$

for any $(\nu_1,\omega_2), (\nu_2,\omega_2) \in (0, 2\pi]^2$.

Asymptotic normality, APC case

Theorem (Lenart 2008)

If

(i) there exists $\delta > 0$ such that $\sup_{t\in\mathbb{Z}} \|X_t\|_{6+3\delta} \leq \Delta < \infty$,

(ii) $w_n = O(n^{\kappa})$ for some $\kappa \in (0, \delta/(4+4\delta)$),

(iii) $\sum_{h=1}^{\infty} h^{2r} \alpha(h)^{\frac{\delta}{2(r+1)+\delta}} < \infty$, where r is the integer such that $r > \max\left\{1+\delta, \frac{1-\kappa}{4\kappa}, \frac{\kappa(1+\delta)}{\delta-2\kappa(1+\delta)}\right\}$,

then

$$\sqrt{\frac{n}{w_n}}\left(\hat{G}_n(\nu,\omega) - P(\nu,\omega)\right) \longrightarrow N(0,\Sigma(\nu,\omega)),$$

where matrix $\Sigma(\nu,\omega)$ can be obtained by previous Theorem.

Asymptotic normality, counting process case

Theorem (Dudek, 2008)
If

(i) The Y process is periodically correlated with period P and its mean function $E(Y(s))$ is bounded away from zero.

(ii) Process Y has the third moment bounded.

(iii) Process Y is α–mixing with $\alpha(k) = o(k^{-3})$.

(iv) Each period P is divided into b parts, where $b = O\left(\sqrt{n}\right)$.

(v) The periodic λ_0 function (with the period length equal to P) and $EY(t)$ fulfill the Lipschitz condition on $[0, P]$

then

$$\sqrt{\frac{n}{b}}\left(\widehat{\lambda}_n(s) - \lambda_0(s)\right) \Rightarrow N\left(0, \frac{\lambda_0(s)}{E(Y(s))}\right).$$

Why resampling?

* APC time series case: too complicated asymptotic covariance matrix
* periodic counting process case: slow convergence, need for simultaneous confidence bands

Subsampling for Fourier coefficient of autocovariance function

Consistency holds for the estimator $\hat{\theta}_n = |\hat{a}_n(\lambda, \tau)|$. Let

$$J_n(x, P) = Prob_P\left(\sqrt{n}(|\hat{a}_n(\lambda, \tau)| - |a(\lambda, \tau)|) \leq x\right).$$

By CLT for $\hat{a}_n(\lambda, \tau)$ and the delta method we have

$$J_n(P) \xrightarrow{d} J(P).$$

We define correspondingly subsampling distribution in the form

$$L_{n,b}(P) = \frac{1}{n-b+1} \sum_{t=1}^{n-b+1} \mathbf{1}\{\sqrt{b}(|\hat{a}_{n,b,t}(\lambda, \tau)| - |\hat{a}_n(\lambda, \tau)|) \leq x\}$$

Subsampling for Fourier coefficient of autocovariance function

Theorem (Lenart, Leśkow, Synowiecki, 2008)

Let $\{X(t) : t \in \mathbb{Z}\}$ be APC time series. Assume that

(i) $b \to \infty$ but $b/n \to 0$,

(ii) $\sup_t E|X(t)|^{4+4\delta} < \infty$,

(iii) $\sum_{k=0}^{\infty} (k+1)^2 \alpha(k)^{\frac{\delta}{4+\delta}} < \infty$,

(iv) the function $V(t, \tau_1, \tau_2, \tau_3) = E\left(X(t)X(t+\tau_1)X(t+\tau_2)X(t+\tau_3)\right)$ is almost periodic.

Then subsampling is consistent, which means that

$$\sup_x |J_n(x, P) - L_{n,b}(x)| \xrightarrow{P} 0.$$

Application of subsampling procedure for PC time series

Testing problem:

$$H_0 : B(\cdot, \tau) \text{ is periodic with period } T_0,$$
$$H_1 : B(\cdot, \tau) \text{ is periodic with period } T_1.$$

Test statistics (Lenart, Leskow, Synowiecki, 2008):

$$U_n(\tau) = \sqrt{n} \left(\sum_{\lambda \in \Lambda_{T_1} \setminus \Lambda_{T_0}} |\hat{a}_n(\lambda, \tau)| \right).$$

Application of subsampling procedure for PC time series

Under H_0:

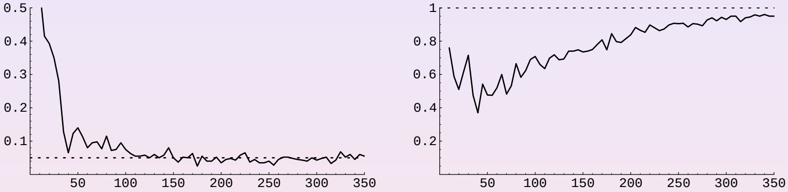
$$U_n(\tau) \xrightarrow{d} J.$$

Under H_1:

$$U_n(\tau) \longrightarrow \infty.$$

Large values of $U_n(\tau)$ suggest that hypothesis H_1 is true. The rejection area is of the form $[c_{1-\alpha}, \infty)$. In order to find $c_{1-\alpha}$ subsampling may be applied.

Application of subsampling procedure for PC time series



(a) Probability of rejection H_0 provided that H_0 is true.

(b) Probability of rejection H_0 provided that H_1 is true.

Figure: Monte Carlo approximations of test errors.

Theorem (Synowiecki, 2007)

Let $\{X_t : t \in \mathbb{Z}\}$ be APC and α-mixing, let $(X_1^*, \ldots, X_n^*)$ be MBB sample, $b \to \infty$ ale $b/n \to 0$. Assume that

(i) $\Lambda = \{\lambda : [0, 2\pi) : M_t(EX_t e^{-i\lambda t}) \neq 0\}$ is finite,

(ii) autocovariance is uniformly summable

(iii) $\sup_{s=1,\ldots,n-b+1} E\left(\frac{1}{\sqrt{b}}\sum_{t=s}^{s+b-1}(X_t - EX_t)\right)^4 < K$

(iv) CLT holds, i.e. $\sqrt{n}\left(\overline{X}_n - M_t(EX_t)\right) \xrightarrow{d} \mathcal{N}(0, \sigma^2)$

Then MBB procedure is consistent, which means that

$$Var^*(\sqrt{n}\,\overline{X}_n^*) \xrightarrow{P} \sigma^2$$

and

$$\sup_{x\in\mathbb{R}} \left| P\left(\sqrt{n}\left(\overline{X}_n - \mu\right) \le x\right) - P^*\left(\sqrt{n}\left(\overline{X}_n^* - E^*\overline{X}_n^*\right) \le x\right)\right| \xrightarrow{p} 0.$$

Consistency of subsampling - APC case, spectral coherence

Theorem (Lenart, 2008)

Under regularity conditions the subsampling confidence intervals for coherence are consistent

$$P\left(\sqrt{n/w_n}\,(|\hat{\gamma}_n(\nu,\omega)| - |\gamma(\nu,\omega)|) \leq c^{\gamma}_{n,b}(1-\alpha)\right) \longrightarrow 1-\alpha,$$

where $b = b(n) \to \infty$, and $b/n \to 0$,

$$c^{\gamma}_{n,b}(1-\alpha) = \inf\{x :\ L^{\gamma}_{n,b}(x) \geq 1-\alpha\}.$$

$$L^{\gamma}_{n,b}(x) = \frac{1}{n-b+1}\sum_{t=1}^{n-b+1} \mathbf{1}\{\sqrt{b/w_b}(|\hat{\gamma}_{n,b,t}(\nu,\omega)| - |\hat{\gamma}_n(\nu,\omega)|) \leq x\}.$$

Consistency of bootstrap - counting process case

Theorem (Dudek, 2008)

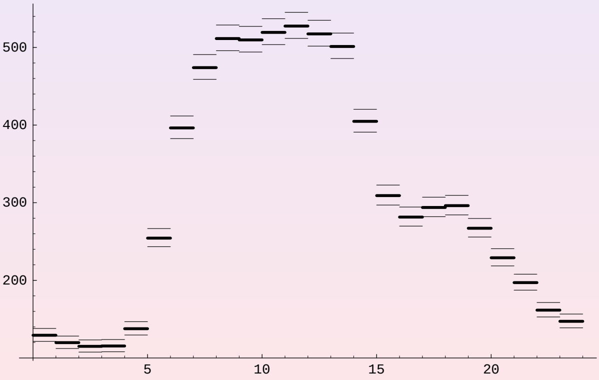
$$\sup_{u \in R} \left| P^* \left(\sqrt{\frac{n}{b}} (\widehat{\lambda}_n^*(s) - \widehat{\lambda}_n(s)) \leq u \right) - P \left(\sqrt{\frac{n}{b}} (\widehat{\lambda}_n(s) - \lambda_0(s)) \leq u \right) \right| = o_P(1),$$

where

$$\widehat{\lambda}_n^*(s) = \frac{\sum_{k=1}^n X_k^*(B_n^s)}{\sum_{k=1}^n \int_{B_n^s} Y_k(u) du} 1_{D_n}(s).$$

Real data example - counting process case

Estimator of the intensity of the number of packets being received by one host together with 90% confidence region:



Future directions of research

* Subsampling - optimal selection of block size
* Resampling in GACS signals
* nonstationary random fields

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Thank you for your attention